Revision 2 (Solutions)

Year 11 Examination

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section Two: Calculator-assumed

Student Number: In figures | | | | | | | | |

In words ______________________________

Teacher name ______________________________

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet

Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section Two: Calculator-assumed **65% (54 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

(a) The point $P(-3, 5)$ is translated by the column vectors $\begin{bmatrix} 2 \\ -7 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$ to $P'(9, -11)$. Determine the values of the constants x and y. (2 marks)

Solution
$\begin{bmatrix} -3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ -7 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -11 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -11 \end{bmatrix} - \begin{bmatrix} -3 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ -7 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -9 \end{bmatrix}$
Specific behaviours
✓ value of x ✓ value of y

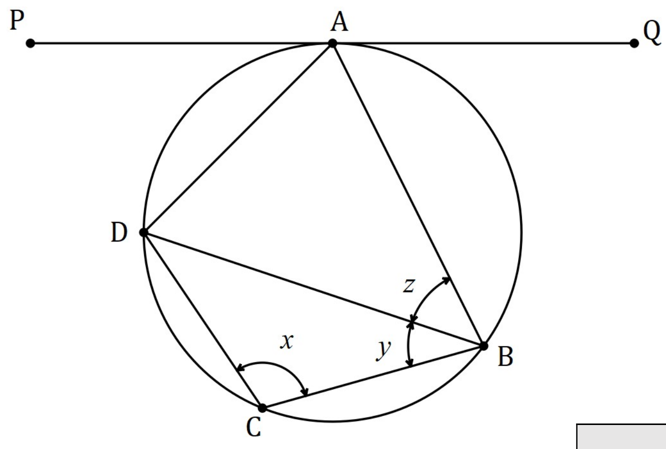
(b) Determine the single matrix that represents, in order, the composition of a rotation of $30°$ anti-clockwise about the origin followed by reflection in the line $y = x$. Express matrix coefficients in exact form. (4 marks)

Solution
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$
Specific behaviours
✓ matrix for rotation   ✓ matrix for reflection   ✓ multiplies in correct order   ✓ correct matrix

Question 2 **(3 marks)**

The diagram below shows four points A, B, C and D lying on the circumference of a circle. The line PQ is a tangent to the circle at A, $\angle BDC = 21°$, $\angle PAD = 35°$ and $\angle QAB = 62°$.

Determine the size of angles x, y and z. (3 marks)



Solution
$x = 180 - (180 - 35 - 62) = 97°$ $y = 180 - 97 - 21 = 62°$ $z = \angle PAD = 35°$
Specific behaviours
✓✓✓ each angle

Question 3 **(9 marks)**

(a) If $\mathbf{p} = 4\mathbf{i} - 2\mathbf{j}$ and $\mathbf{q} = 3\mathbf{i} + 2\mathbf{j}$ determine

(i) the angle between the directions of $\mathbf{p}$ and $\mathbf{q}$, to the nearest tenth of a degree. (2 marks)

Solution
Using CAS angle is $60.255 \approx 60.3°$ (1dp)
Specific behaviours
✓ states angle ✓ rounds correctly

(ii) the scalar projection of $\mathbf{q}$ on $\mathbf{p}$. (2 marks)

Solution
$\sqrt{3^2 + 2^2} \times \cos(60.255)$ ≈ 1.79 Exact: $\frac{4\sqrt{5}}{5}$
Specific behaviours
✓ expression for projection ✓ states value

(b) The vector $21\mathbf{i} + 5a\mathbf{j}$ has a magnitude of 29 and is perpendicular to the vector $4\mathbf{i} - 2b\mathbf{j}$. Determine the values of the constants a and b, where $a < b$. (5 marks)

Solution
$21^2 + (5a)^2 = 29^2$ $a = \pm 4$ $(21)(4) + (5a)(-2b) = 0$ $84 - 10(\pm 4)b = 0$ $b = \pm\frac{21}{10}$ $a = -4, \qquad b = -\frac{21}{10}$
Specific behaviours
✓ uses magnitude to form equation ✓ calculate values of a ✓ uses dot product to form equation ✓ calculate values of b ✓ chooses correct pairing

Question 4 **(6 marks)**

(a) Prove that $\sin 3A = 3\sin A - 4\sin^3 A$. (4 marks)

Solution
$LHS = \sin 3A$ $= \sin(A + 2A)$ $= \sin A\cos 2A + \cos A\sin 2A$ $= \sin A\,(1 - 2\sin^2 A) + \cos A\,(2\sin A\cos A)$ $= \sin A - 2\sin^3 A + 2\sin A\,(1 - \sin^2 A)$ $= 3\sin A - 4\sin^3 A$ $= RHS$
Specific behaviours
✓ expands sum of A and 2A ✓ uses double angle identity for cos2A ✓ uses double angle identity for sin2A ✓ expands and simplifies

(b) Hence, or otherwise, solve $3\sin A - 4\sin^3 A = \frac{1}{2}, 0 \le A \le \frac{\pi}{3}$. (2 marks)

Solution
$\sin 3A = \dfrac{1}{2}$ $3A = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ $A = \dfrac{\pi}{18}, \dfrac{5\pi}{18}$
Specific behaviours
✓ one correct solution ✓ all solutions within required domain

Question 5 **(9 marks)**

Triangle ABC has vertices $A(1,2)$, $B(4,-1)$ and $C(5,3)$.

(a) The vertices ABC are transformed to $A'B'C'$ using matrix $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$. Write down the new coordinates of the vertices and describe the transformation. (4 marks)

Solution
$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 4 & 5\\ 2 & -1 & 3\end{bmatrix} = \begin{bmatrix}-1 & -4 & -5\\ 2 & -1 & 3\end{bmatrix}$ $A'(-1,2), B'(-4,-1), C'(-5,3)$ Transformation is a reflection in the line $x = 0$.
Specific behaviours
✓ matrix product ✓ writes coordinates of vertices ✓ states reflection ✓ states equation of line of reflection

(b) The vertices ABC are transformed to $A''B''C''$ by matrix M so that the new coordinates of the vertices are $A''(-4,3)$, $B''(2,12)$ and $C''(-6,15)$.

(i) Determine the transformation matrix M. (3 marks)

Solution
$M\begin{bmatrix}1 & 4\\ 2 & -1\end{bmatrix} = \begin{bmatrix}-4 & 2\\ 3 & 12\end{bmatrix}$ $M = \begin{bmatrix}-4 & 2\\ 3 & 12\end{bmatrix}\begin{bmatrix}1 & 4\\ 2 & -1\end{bmatrix}^{-1}$ $M = \begin{bmatrix}0 & -2\\ 3 & 0\end{bmatrix}$
Specific behaviours
✓ writes matrix equation ✓ post-multiplies by inverse ✓ determines M

(ii) If the area of triangle ABC is k square units, express the area of triangle $A''B''C''$ in terms of k. (2 marks)

Solution
$\lvert M\rvert = 6$ $\text{New area} = 6k$
Specific behaviours
✓ determinant of M ✓ expresses area

Question 6 **(7 marks)**

(a) How many numbers must be chosen from the set of integers between 1 and 2017 inclusive to be certain that one of the numbers chosen is a multiple of 10. (3 marks)

Solution
201 multiples of 10 between 1 to 2017. Require $2017 - 201 = 1816$ pigeonholes. Hence must choose $1\,817$ integers.
Specific behaviours
✓ states # of multiples in set ✓ one pigeonhole for every non-multiple ✓ uses pigeonhole principle to add one

(b) A number is formed using four different digits chosen from those in the number 23 814. Determine how many different numbers can be formed that are

(i) even. (1 mark)

Solution
$n(A) = 3 \times 4 \times 3 \times 2 = 72$
Specific behaviours
✓ states number

(ii) greater than 8 000. (1 mark)

Solution
$n(B) = 1 \times 4 \times 3 \times 2 = 24$
Specific behaviours
✓ states number

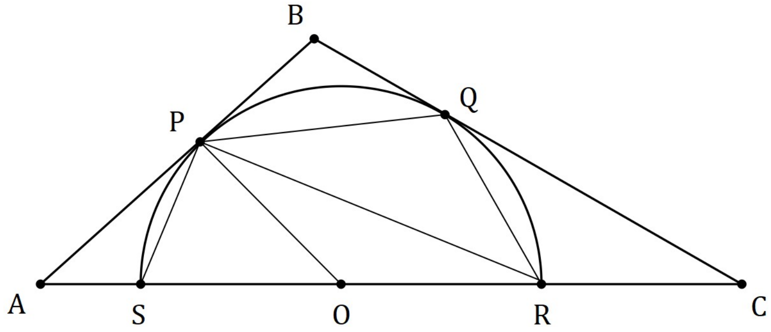
(iii) even or greater than 8 000. (2 marks)

Solution
$n(A \cap B) = 1 \times 2 \times 3 \times 2 = 12$ $n(A \cup B) = 72 + 24 - 12 = 84$ 84 numbers
Specific behaviours
✓ calculates number even and greater than 8 000 ✓ states number

Question 7 **(6 marks)**

The diagram shows a semi-circle, with diameter SR and centre O, circumscribed by triangle ABC, in which $\angle BAC = 42°$ and $\angle BCA = 32°$.

Determine, with reasons, the size of angles $\angle PRO$ and $\angle PRQ$.



Solution
$\angle AOP = 90 - 42 = 48°$ ($\angle APO = 90°$, tangent-radius) $\angle PRO = 48 \div 2 = 24°$ (centre $\angle$=2×circumference $\angle$) $\angle COQ = 90 - 32 = 58°$ (as $\angle AOP$) $\angle ORQ = (180 - 58) \div 2 = 61$ (ΔOQR isosceles) $\angle PRQ = 61 - 24 = 37°$

Specific behaviours
✓ $\angle AOP$ ✓ $\angle PRO$ ✓ reasoning ✓ $\angle ORQ$ ✓ $\angle PRQ$ ✓ reasoning

Question 8 **(8 marks)**

The sum of the first n terms of the sequence $2 + 8 + 14 + 20 + \cdots + (6n - 4)$ is $n(3n - 1)$.

(a) Show that this statement is true when $n = 5$. (2 marks)

Solution
$LHS = 2 + 8 + 14 + 20 + 26 = 70$ $RHS = 5(3(5) - 1) = 5 \times 14 = 70$ Hence statement true
Specific behaviours
✓ shows sum of terms for LHS ✓ shows substitution in RHS and states true

(b) Use mathematical induction to prove the statement is true for $n \in \mathbb{Z}, n \geq 5$. (6 marks)

Solution
Assume statement true when $n = k$: $2 + 8 + 14 + 20 + \cdots + (6k - 4) = k(3k - 1)$ When $n = k + 1$: $2 + 8 + 14 + 20 + \cdots + (6k - 4) + (6k - 4 + 6) = k(3k - 1) + (6k - 4 + 6)$ $= 3k^2 + 5k + 2$ $= (k + 1)(3k + 2)$ $= (k + 1)(3(k + 1) - 1)$ $= n(3n - 1)$ when $n = k + 1$ The statement is true for $n = 5$ and by induction, the truth when $n = k$ implies the truth when $n = k + 1$ and hence the statement is true for $n \geq 5$.
Specific behaviours
✓ assumed true for $n = k$ ✓ adds next term to both sides ✓ simplifies RHS ✓ factors k+1 out of RHS ✓ indicates true for $n = k + 1$ ✓ summary statement including truth of $n = 5$